Exercise 30

Find the orthogonal trajectories of the family of curves. Use a graphing device to draw several members of each family on a common screen.

$$y^2 = kx^3$$

Solution

To find the orthogonal trajectories, we have to solve for $y'(x)$, set $y'_\perp$ equal to the negative reciprocal, and then solve for $y_\perp$. Start by differentiating both sides of the given equation with respect to x.

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(kx^3\right)$$
$$2y\frac{dy}{dx} = 3kx^2$$

Divide both sides by $2y$.

$$\frac{dy}{dx} = \frac{3kx^2}{2y}$$

Solve the original equation for k,

$$k = \frac{y^2}{x^3},$$

and plug the expression into the equation.

$$\frac{dy}{dx} = \frac{3x^2}{2y}\frac{y^2}{x^3} = \frac{3y}{2x}$$

Here is where we introduce $y_\perp$.

$$\frac{dy_\perp}{dx} = -\frac{2x}{3y_\perp}$$

Since this equation is separable, we can solve for $y_\perp$ by bringing all terms with $y_\perp$ to the left and all constants and terms with x to the right and then integrating both sides.

$$y_\perp\, dy_\perp = -\frac{2}{3}x\, dx$$
$$\int y_\perp\, dy_\perp = \int -\frac{2}{3}x\, dx$$
$$\frac{1}{2}y_\perp^2 = -\frac{1}{3}x^2 + C$$

Multiply both sides by 2.

$$y_\perp^2 = -\frac{2}{3}x^2 + 2C$$

Take the square root of both sides and let $A = 2C$.

$$y_\perp(x) = \pm\sqrt{-\frac{2}{3}x^2 + A}$$

This is the family of curves orthogonal to $y^2 = kx^3$.

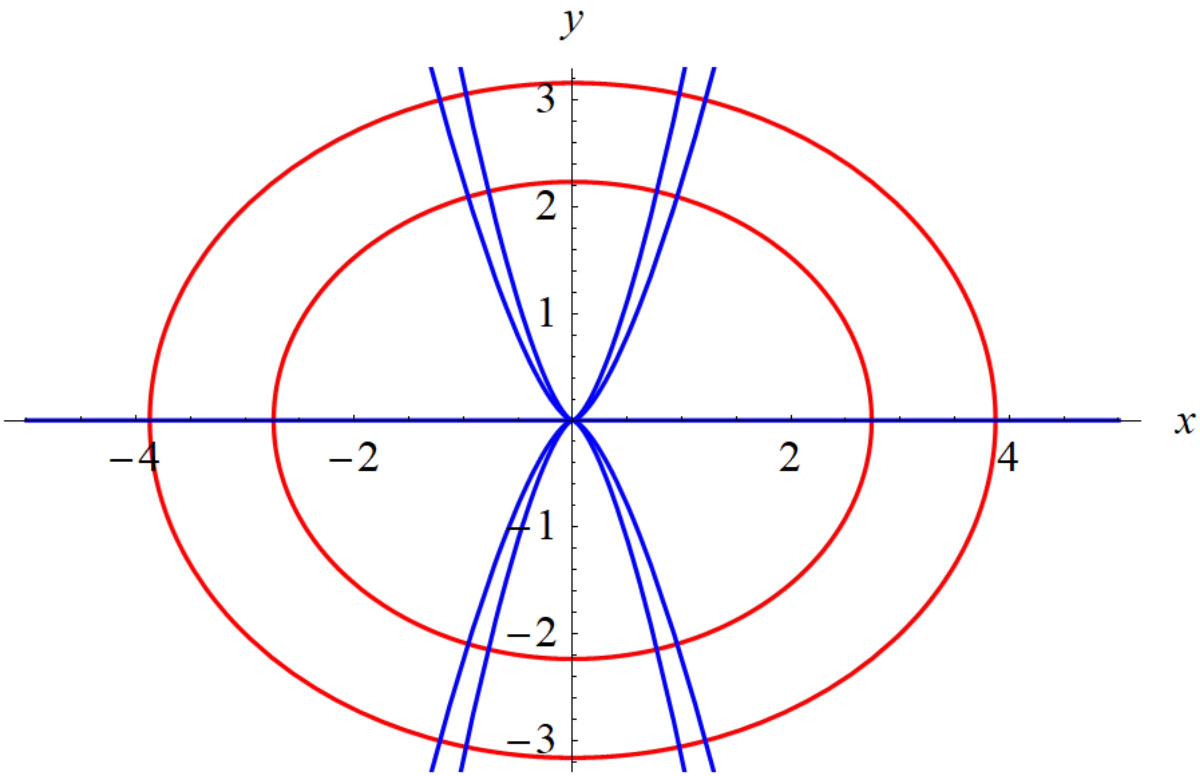

Figure 1: Plot of $y^2 = kx^3$ in blue ($k = 0, \pm 5, \pm 10$) and the orthogonal trajectories $y_\perp$ in red ($A = 5, 10$).